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B.Sc-I

Mathematics Hons: Paper-I

GROUP A: SET THEORY

Contents :→ sets, subsets, Power set, union, intersection.

Definition :→ A set is a well defined collection of distinct objects.

Example :→

1. $\mathbb{N} = \{1, 2, 3, \ldots\ldots\}$, the set of all natural numbers.

2. $\mathbb{Z} = \{0, \pm1, \pm2, \ldots\ldots\}$, the set of all integers,

3. $\mathbb{Q} = \{\frac{p}{q} : p, q \in \mathbb{Z}, q \neq 0\}$, the set of all rational numbers.

Definition :→ Let A and B be two sets. If $x \in A \Rightarrow x \in B$, then A is said to be subset of B.

That means that each element of A is an element of B. It is denoted by $A \subseteq B$.

Example :- Let $A = \{1, 2, 3, 4\}$ & $B = \{1, 2, 3, 4, 5\}$ then $A \subseteq B$.

Equality of sets

Two sets A and B are said to be equal if they have the same elements.

In other words, $A = B$ iff $A \subseteq B$ & $B \subseteq A$.

Power set :→ For a set S, the power set of S is defined to be the family of all subsets of S. It is denoted by P(S). Thus

$$P(S) = \{A : A \subseteq S\}$$

Example :- Let $A = \{a, b, c\}$. Then

$$P(A) = \{\phi, \{a\}, \{b\}, \{c\}, \{a,b\}, \{a,c\}, \{b,c\}, \{a,b,c\}\}$$

THEOREM :→ A set, containing n elements, has exactly 2^n subsets.

Proof :→ Let $A = \{a_1, a_2, \ldots, a_n\}$ containing n elements.

Clearly the null set ϕ is a subset of the set A.

Since Number of subsets of A containing one elements viz., $\{a_1\}, \{a_2\}, \ldots, \{a_n\} = {}^nC_1$

Number of subsets of A containing two elements $= {}^nC_2$

Similarly, there will be ${}^nC_3$ subsets of A each containing three elements & so on.

$\therefore$ The number of subsets of A containing n elements $= {}^nC_n$

Hence the total number of subsets of $A = {}^nC_0 + {}^nC_1 + {}^nC_2 + \cdots\cdots + {}^nC_n$

$= (1+1)^n = 2^n$

proved.

Example ① If $A = \phi$. Then

$P(A) = \{\phi\}$

Example ② If the set A contains n elements, then show that $P(A)$ contains 2^n elements.

Union and Intersection of Sets:

Let A and B be two sets. The Union of A and B is the set defined by

$A \cup B = \{x : x \in A \text{ or } x \in B\}$

That is, $x \in A \cup B$ if and only if $x \in A$ or $x \in B$.

The intersection of A and B is the set defined by

$A \cap B = \{x : x \in A \text{ and } x \in B\}$

That is, $x \in A \cap B$ if and only if $x \in A$ and $x \in B$.

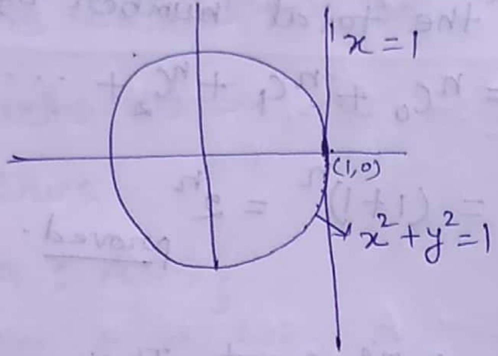
~~Exercise~~

Exercise 1. If $A \subseteq B$, then find $A \cup B$ and $A \cap B$.

Solution :- $A \cup B = B$ & $A \cap B = A$

Ex.2 :→ Let $A := \{(x,y) \in \mathbb{R}^2 : x^2+y^2=1\}$ and $B := \{(x,y) \in \mathbb{R}^2 : x = 1\}$. Find $A \cap B$.

Solution :→



$\therefore A \cap B = \{(1,0)\}$

Ex:3 :→ Let A be the set of $n \times n$ real symmetric matrices, & B the set of $n \times n$ real skew-symmetric matrices. Find $A \cap B$.

(TRY YOURSELF)

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MATHEMATICS HONS : Paper-I

Group A : SET THEORY

Contents :→ Complement of a set, De Morgan's Law.

Complement of a set :→ Whenever we consider a set, its elements are chosen from some set which we call the universe or the universal set.

If A is a part of the universe U, that is, if $A \subseteq U$, then the complement of A (in U), written as A^c, is defined to be

$$A^c := U|A = \{x \in U : x \notin A\}$$

Example :→ Let $U = \{a, b, c, d, e\}$, & $A = \{a, e\}$,
Then $A^c = U|A = \{x \in U : x \notin A\}$

$$\therefore A^c = \{b, c, d\}$$

THEOREM: → (De Morgan's Law).

Let A and B be subsets of a universal set U. Then

ⓐ $(A \cap B)^c = A^c \cup B^c$

ⓑ $(A \cup B)^c = A^c \cap B^c$

Proof ⓐ: → We need to show that $(A \cap B)^c \subseteq A^c \cup B^c$ and $A^c \cup B^c \subseteq (A \cap B)^c$.

Let $x \in (A \cap B)^c$

$\Rightarrow \{x \in U : x \notin (A \cap B)\}$

$\cancel{\{x \in U : x \notin A \text{ and } x \notin B\}}$

$\Rightarrow \{x \in U : x \notin A \text{ or } x \notin B \text{ or } x \notin A \text{ and } B\}$

$\Rightarrow \{x \in U : x \in A^c \text{ or } x \in B^c\}$

$\Rightarrow \{x \in U : x \in A^c\} \text{ or } \{x \in U : x \in B^c\}$

$\cancel{x \in A^c \cup B^c} \Rightarrow x \in (A^c \cup B^c)$

$\therefore (A \cap B)^c \subseteq (A^c \cup B^c)$ —— ①

Let $y \in (A^c \cup B^c)$

$\cancel{y \in A^c \text{ or } y \in B^c}$

$\Rightarrow \{y \in U : y \in A^c \text{ or } y \in B^c\}$

$\Rightarrow \{y \in U : y \notin A \text{ and } y \notin B\}$

$\Rightarrow \{y \in U : y \notin (A \cap B)\}$

$\Rightarrow \{y \in U : y \in (A \cap B)^c\}$

$\Rightarrow y \in (A \cap B)^c$

$\therefore (A^c \cup B^c) \subseteq (A \cap B)^c$ —— ②

From ① & ② we get

$$(A \cap B)^c = A^c \cup B^c$$ proved.

Proof ⓑ :→ We need to show that

$$(A \cup B)^c \subseteq A^c \cap B^c \quad \& \quad A^c \cap B^c \subseteq (A \cup B)^c$$

Let $x \in (A \cup B)^c$

$\Rightarrow \{x \in U : x \in (A \cup B)^c\}$

$\Rightarrow \{x \in U : x \notin (A \cup B)\}$

$\Rightarrow \{x \in U : x \notin A \text{ and } x \notin B\}$

$\Rightarrow \{x \in U : x \in A^c \text{ and } x \in B^c\}$

$\Rightarrow \{x \in U : x \in A^c \cap B^c\}$

$\Rightarrow x \in A^c \cap B^c$

$\therefore (A \cup B)^c \subseteq A^c \cap B^c$ ——— ③

Let $y \in (A^c \cap B^c)$

$\Rightarrow \{y \in U : y \in A^c \text{ and } y \in B^c\}$

$\Rightarrow \{y \in U : y \notin A \text{ or } y \notin B\}$

$\Rightarrow \{y \in U : y \notin A \cup B\}$

$\Rightarrow \{y \in U : y \in (A \cup B)^c\}$

$\Rightarrow y \in (A \cup B)^c$

$\therefore A^c \cap B^c \subseteq (A \cup B)^c$ ——— ④

From equation ③ & 4

$$(A \cup B)^c = A^c \cap B^c$$ proved.

Family of sets :→

Suppose Λ is a non-empty set, and for each $\alpha \in \Lambda$ there is a set A_α. Then, we have a family of sets indexed by Λ which is written as

$$\{A_\alpha : \alpha \in \Lambda\}$$

Here the set Λ is called the index set for the family.

Union of family of sets :→

Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of sets indexed by a non-empty set Λ. Let us assume that each of the sets in the family is a subset of some universal set U.

The union of family of sets is defined as

$$\bigcup_{\alpha \in \Lambda} F_\alpha = \{x \in U : \exists\, \alpha \in \Lambda \text{ such that } x \in F_\alpha\}$$

Intersection of family of sets :→

Let $\{F_\alpha : \alpha \in \Lambda\}$ be a family of sets indexed by a non-empty set Λ. Let us assume that each of these sets is a subset of some universal set U. Then the intersection of this family is defined by

$$\bigcap_{\alpha \in \Lambda} F_\alpha = \{x \in U : \forall\, \alpha \in \Lambda,\ x \in F_\alpha\}$$

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Contents:- Generalised De Morgan's Law,
Generalised distributive law,
Generalised Associative law.

Remark:→ Let $\{A_i : i \in I\}$ be an indexed family of subsets of the universal set U. Then

$\bigcup_{i\in I} A_i = \{x \in U \mid x \in A_i$ for at least one $i \in I\}$ &

$\bigcap_{i\in I} A_i = \{x \in U \mid x \in A_i,$ for all $i \in I\}$

Generalised De Morgan's law:

If $\{A_i ; i \in I\}$ be an index family of subsets of the universal set U. Then

(i) $\left(\bigcup_i A_i\right)^c = \bigcap_i A_i^c$

(ii) $\left(\bigcap_i A_i\right)^c = \bigcup_i A_i^c$

Proof (i):→ Let x be any element of U (universal set)

Let $x \in \left(\bigcup_i A_i\right)^c$

$\Leftrightarrow x \notin \bigcup_i A_i$

$\Leftrightarrow x \notin A_i$, for any $i \in I$

$\Leftrightarrow x \in A_i^c$ for all $i \in I$

$\Leftrightarrow x \in \bigcap_i A_i^c$

$\therefore (\bigcup_i A_i)^c \subseteq \bigcap_i A_i^c$

and $\bigcap_i A_i^c \subseteq (\bigcup_i A_i)^c$

$\therefore (\bigcup_i A_i)^c = \bigcap_i A_i^c$ proved.

Proof(ii) :→ Let x be any element of the universal set U.

Let $x \in (\bigcap_i A_i)^c \Leftrightarrow x \notin \bigcap_i A_i$

$\Leftrightarrow x \notin A_i$ for at least one $i \in I$.

$\Leftrightarrow x \in A_i^c$ for at least one $i \in I$.

$\Leftrightarrow x \in \bigcup_i A_i^c$

$\therefore (\bigcap_i A_i)^c \subseteq \bigcup_i A_i^c$

and $\bigcup_i A_i^c \subseteq (\bigcap_i A_i)^c$

$\therefore (\bigcap_i A_i)^c = \bigcup_i A_i^c$ proved.

Generalised distributive law :→ If $\{A_i : i \in I\}$ be an indexed family of subsets of the universal set U and $B \subseteq U$ then

(i) $B \cup (\bigcap_i A_i) = \bigcap_i (B \cup A_i)$

(ii) $B \cap (\bigcup_i A_i) = \bigcup_i (B \cap A_i)$

Proof(i): → Let x be any element of the universal set U.

$B \cup (\bigcap_i A_i)$

$= \{x \in U \mid x \in B \text{ or } x \in (\bigcap_i A_i)\}$

$= \{x \in U \mid x \in B \text{ or } (x \in A_i \text{ for each } i \in I)\}$

$= \{x \in U \mid (x \in B \text{ or } x \in A_i) \text{ for each } i \in I\}$

$= \{x \in U \mid x \in (B \cup A_i) \text{ for each } i \in I\}$

$= \bigcap_i (B \cup A_i)$

proved.

Proof(ii): → $B \cap (\bigcup_i A_i)$

$= \{x \in U \mid x \in B \text{ and } x \in (\bigcup_i A_i)\}$

$= \{x \in U \mid x \in B \text{ and } (x \in A_i \text{ for at least one } i \in I)\}$

$= \{x \in U \mid (x \in B \text{ and } x \in A_i) \text{ for at least one } i \in I\}$

$= \{x \in U \mid x \in (B \cap A_i) \text{ for at least one } i \in I\}$

$= \bigcup_i (B \cap A_i)$

proved.

Generalised Associative law: → If $\{A_i : i \in I\}$ be an indexed family of subset of U and $B \subseteq U$ then

(i) $B \cup (\bigcup_i A_i) = \bigcup_i (B \cup A_i)$

(ii) $B \cap (\bigcap_i A_i) = \bigcap_i (B \cap A_i)$

Proof(i): →

$B \cup (\bigcup_i A_i)$

$= \{x \in U \mid x \in B \text{ or } x \in (\bigcup_i A_i)\}$

$= \{x \in U \mid x \in B \text{ or } (\exists\, i_0 \in I \text{ s.t. } x \in A_{i_0})\}$

$= \{x \in U \mid \exists\, i_0 \in I \text{ such that } (x \in B \text{ or } x \in A_{i_0})\}$

$= \{x \in U \mid \exists\, i_0 \in I \text{ such that } (x \in B \cup A_{i_0})\}$

$= \bigcup_i (B \cup A_i)$

proved

Proof(ii): → $B \cap (\bigcap_i A_i)$

$= \{x \in U \mid x \in B \text{ and } x \in (\bigcap_i A_i)\}$

$= \{x \in U \mid x \in B \text{ and } (x \in A_i \text{ for each } i \in I)\}$

$= \{x \in U \mid (x \in B \text{ and } x \in A_i) \text{ for each } i \in I\}$

$= \{x \in U \mid x \in (B \cap A_i) \text{ for each } i \in I\}$

$= \bigcap_i (B \cap A_i)$

proved.